

# TAMARKIN'S PROOF OF KONTSEVICH FORMALITY THEOREM

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## 1. INTRODUCTION

1.1. This is an extended version of lectures given at Luminy colloquium “Quantification par déformation” held at December, 1999.

In this note we explain Tamarkin’s proof [T] of the following affine algebraic version of Kontsevich’s formality theorem.

1.2. **Theorem.** *Let  $A$  be a polynomial algebra over a field  $k$  of characteristic zero and let  $\mathcal{C} = C^*(A; A)$  be the cohomological Hochschild complex of  $A$  with coefficients at  $A$ . The dg Lie algebra  $\mathcal{C}[1]$  is formal, that is  $\mathcal{C}[1]$  is isomorphic in the homotopy category of dg Lie algebras to its cohomology.*

Our sources are the original Tamarkin’s paper [T] and the recent preprint of Tamarkin-Tsygan [TT] where a simplification of the original proof is sketched.

We tried to provide all necessary details that were sometimes difficult to find in [T].

1.3. In the first part (Sections 2–3) we review some basic facts on operads and Koszul operads. In Section 4 we study formality of algebras over a Koszul operad. Following Halperin-Stasheff [HS], we call a graded algebra  $H$  over a Koszul operad  $\mathcal{O}$  *intrinsically formal* if any dg  $\mathcal{O}$ -algebra with cohomology isomorphic to  $H$  is formal. We prove Theorem 4.1.3 which gives a sufficient condition of intrinsic formality of a graded algebra over a Koszul operad in terms of its cohomology.

In Section 5 we calculate the cohomology of Gerstenhaber algebra  $H(\mathcal{C})$ ,  $\mathcal{C}$  being the Hochschild complex of a smooth  $k$ -algebra. The calculation shows that  $H(\mathcal{C})$  is intrinsically formal for a polynomial algebra. This proves that Kontsevich’s formality theorem follows from a version of Deligne’s conjecture 5.3.3 on the existence of homotopy Gerstenhaber algebra structure on the Hochschild complex. Theorem 5.3.3 is proven in Sections 6 and 7.

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## 2. BASIC DEFINITIONS

In this section we recall the basic definitions of operads and operad algebras. The most convenient reference here is [GJ], ch. 1.

**2.1. Operads.** Let  $\mathbf{Vect}$  be the category of vector spaces over a field  $k$  of characteristic zero.

By definition, an  $\mathbb{S}$ -object in  $\mathbf{Vect}$  is a collection  $X = \{X(n)\}$ ,  $n \geq 0$ , of objects of  $\mathbf{Vect}$  endowed with a right action of the symmetric groups  $S_n$ . The category of  $\mathbb{S}$ -objects in  $\mathbf{Vect}$  admits a (non-symmetric) monoidal structure defined as follows.

Any  $\mathbb{S}$ -object  $X$  defines a functor  $\mathcal{S}(X)$  (Schur functor) on  $\mathbf{Vect}$  by the formula

$$(1) \quad \mathcal{S}(X) : V \mapsto \bigoplus X(n) \otimes_{S_n} V^{\otimes n}.$$

The monoidal operation on the category of  $\mathbb{S}$ -vector spaces is uniquely defined by the property

$$\mathcal{S}(X \circ Y) = \mathcal{S}(X) \circ \mathcal{S}(Y).$$

**2.1.1. Definition.** An operad  $\mathcal{O} = \{\mathcal{O}(n)\}$  in  $\mathbf{Vect}$  is a monoid in the category of  $\mathbb{S}$ -vector spaces. The category of operads in  $\mathbf{Vect}$  is denoted  $\mathbf{Op}(\mathbf{Vect})$ .

In more conventional terms, an operad is an  $\mathbb{S}$ -vector space  $\{\mathcal{O}(n)\}$  endowed with equivariant operations

$$(2) \quad \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \dots \otimes \mathcal{O}(m_n) \rightarrow \mathcal{O}(\sum m_i)$$

and with a unit element  $1 \in \mathcal{O}(1)$  satisfying natural associativity and unit conditions.

**2.1.2.** For any vector space  $V$  one defines an operad  $\text{Endop}(V)$  to be a  $\mathbb{S}$ -vector space

$$n \mapsto \text{Hom}(V^{\otimes n}, V)$$

with the obvious composition and action of the symmetric groups.

**2.1.3. Definition.** An algebra  $A$  over an operad  $\mathcal{O}$  is a map of operads

$$\mathcal{O} \rightarrow \text{Endop}(A).$$

In other terms, an  $\mathcal{O}$ -algebra structure on  $A$  is given by a collection of  $S_n$ -equivariant maps

$$\mathcal{O}(n) \otimes A^{\otimes n} \rightarrow A$$

satisfying natural associativity and unit properties.

**2.1.4. Examples.** There are operads **ASS**, **COM**, **LIE** such that corresponding algebras are associative, commutative and Lie algebras respectively.

**2.2. Other tensor categories.** The definitions of the previous subsection make sense in any tensor (= monoidal symmetric) category  $\mathcal{A}$ . The following cases will be of a special interest for us.

2.2.1.  $\mathcal{A} = \mathbf{Vectgr}$  — the category of  $\mathbb{Z}$ -graded vector spaces. The commutativity constraint  $X \otimes Y \xrightarrow{\sim} Y \otimes X$  is defined by the standard formula

$$(3) \quad x \otimes y \mapsto (-1)^{|x||y|} y \otimes x.$$

2.2.2.  $\mathcal{A} = C(k)$  — the category of complexes over  $k$ . The commutativity constraint in this case is given by the same formula (3).

2.2.3. Let  $\mathcal{O} \in \mathbf{Op}(\mathcal{A})$  for a tensor category  $\mathcal{A}$  and let  $\alpha : \mathcal{A} \rightarrow \mathcal{B}$  be a tensor functor. Then  $\alpha(\mathcal{O})$  is an operad over  $\mathcal{B}$ . This obvious construction allows one, for example, to consider graded or dg Lie algebras as algebras over the operad LIE in  $\mathbf{Vectgr}$  or in  $C(k)$  respectively.

2.2.4. Denote  $k[n]$  to be a standard one-dimensional vector space concentrated at degree  $-n$ . One defines the  $n$ -shift functor  $X \mapsto X[n]$  by the formula

$$X[n] = k[n] \otimes X.$$

This formula makes sense both in  $\mathbf{Vectgr}$  and in  $C(k)$ .

Let  $\mathcal{O}$  be an operad in  $\mathbf{Vectgr}$  or  $C(k)$ . There is a uniquely defined operad  $\mathcal{O}\{m\}$  such that a  $\mathcal{O}\{m\}$ -algebra structure on  $X$  is equivalent to a  $\mathcal{O}$ -algebra structure on  $X[m]$ . One has

$$\mathcal{O}\{m\}(n) = \Lambda_n^{\otimes m} \otimes \mathcal{O}(n)$$

where  $\Lambda_n$  denotes the graded vector space (or complex)  $k[n-1]$  endowed with the sign representation of the symmetric group  $S_n$ .

### 2.3. Free algebras and free operads.

2.3.1. Let  $\mathcal{O}$  be an operad over a tensor category  $\mathcal{A}$ . Let  $V$  be an  $\mathbb{S}$ -object in  $\mathcal{A}$ . The free  $\mathcal{O}$ -algebra generated by  $V$  is defined to be

$$(4) \quad \mathbb{F}_{\mathcal{O}}(V) = \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{S_n} V^{\otimes n}$$

with a canonical  $\mathcal{O}$ -algebra structure.

2.3.2. Let  $X$  be an  $\mathbb{S}$ -object in  $\mathcal{A}$ . The forgetful functor from the category of operads to the category of  $\mathbb{S}$ -objects in  $\mathcal{A}$  admits a left adjoint *free operad functor*. Free operad  $\mathbb{T}(X)$  generated by  $X$  has an explicit description as a direct sum over trees (see [GJ], 1.4).

### 2.4. Cooperads and coalgebras.

2.4.1. The notions of operad and algebra can be dualized. Let  $\mathcal{C}$  be a cooperad. A  $\mathcal{C}$ -coalgebra  $X$  is called *nilpotent* if

$$(5) \quad X = \cup_n \text{Ker}(X \rightarrow \mathcal{C}(n) \otimes X^{\otimes n}).$$

From now on all coalgebras will be supposed to be nilpotent. We define  $\text{Coalg}(\mathcal{C})$  to be the category of nilpotent  $\mathcal{C}$ -coalgebras. If  $V$  is an  $\mathbb{S}$ -object in  $\mathcal{A}$ , the cofree (nilpotent) cocoalgebra cogenerated by  $V$  is defined to be

$$(6) \quad \mathbb{F}_{\mathcal{C}}^*(V) = \bigoplus_{n \geq 0} (\mathcal{C}(n) \otimes V^{\otimes n})^{S_n}.$$

Let  $X$  be a  $\mathcal{C}$ -coalgebra and  $V$  be an  $\mathbb{S}$ -object. Any map  $X \rightarrow V$  of  $\mathbb{S}$ -objects defines canonically a map of  $\mathcal{C}$ -coalgebras  $X \rightarrow \mathbb{F}_{\mathcal{C}}^*(V)$ .  $V$  is called an  $\mathbb{S}$ -object of *cogenerators* if the above map is injective.

Cofree cooperad cogenerated by  $V$  is denoted  $\mathbb{T}^*(V)$ . It is isomorphic to  $\mathbb{T}(V)$  as an  $\mathbb{S}$ -object. However, we prefer to have a different notation to stress that this is a cooperad.

2.4.2. Let  $\mathcal{O} \in \text{Op}(\text{Vectgr})$  be an operad such that  $\mathcal{O}(n)$  are all finite dimensional. Then the collection  $\{\mathcal{O}(n)^*\}$  admits an obvious structure of cooperad. This cooperad is denoted by  $\mathcal{O}^*$ . Coalgebras over  $\mathcal{O}^*$  are sometimes called  $\mathcal{O}$ -coalgebras. In the same style, we will sometimes write  $\mathbb{F}_{\mathcal{O}}^*(V)$  instead of  $\mathbb{F}_{\mathcal{O}^*}^*(V)$ . Thus,  $\text{COM}$ -coalgebras are just cocommutative coalgebras,  $\text{LIE}$ -coalgebras are Lie coalgebras, etc.

### 3. KOSZUL DUALITY

#### 3.1. Quadratic operads and quadratic duals.

3.1.1. **Definition.** An operad  $\mathcal{O}$  of graded vector spaces is called *quadratic* if it is generated (as operad) by  $\mathcal{O}(2)$  and has only relations of valence 3.

The latter condition means the following. Let  $V$  be the  $\mathbb{S}$ -object in  $\text{Vectgr}$  defined by the properties  $V(2) = \mathcal{O}(2)$ ,  $V(n) = 0$  for  $n \neq 2$ . Since  $\mathcal{O}$  is generated by its binary operations, the natural map  $\mathbb{T}(V) \rightarrow \mathcal{O}$  is surjective.

The operad  $\mathcal{O}$  is quadratic if the kernel of this map is generated (as an ideal in an operad) by an  $S_3$ -invariant subspace  $R \subseteq \mathbb{T}(V)(3)$ .

Note that  $\mathbb{T}(V)(3) = \text{Ind}_{S_2}^{S_3}(V \otimes V)$  where  $S_2$  acts on the tensor product through the trivial action on the first factor.

A quadratic operad  $\mathcal{O}$  with generators  $V$  and relations  $R$  can be described as the pushout in the category of operads

$$\begin{array}{ccc} \mathbb{T}(R) & \longrightarrow & \mathbb{T}(V) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathcal{O} \end{array}$$

where  $*$  denotes the trivial operad

$$*(1) = k, \quad *(n) = 0 \text{ for } n \neq 1.$$

Dually, let  $V$  be a graded vector space endowed with an action of  $S_2$  and let  $R$  be an  $S_3$ -invariant subspace of  $\mathbb{T}^*(V)(3)$ . Denote  $Q = \mathbb{T}^*(V)(3)/R$ . Then a quadratic cooperad  $\mathcal{C}$  cogenerated by  $V$  with co-relations  $R$  is defined as the pullback

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathbb{T}^*(V) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{T}^*(Q) \end{array}$$

**3.1.2. Definition.** rewrite!!! 1. Let  $\mathcal{O}$  be a quadratic operad with  $V = \mathcal{O}(2)$  and the space of relations  $R$ . The dual cooperad  $\mathcal{O}^\perp$  is cogenerated by the space  $V[1]$  with co-relations  $\mathcal{O}^\perp(3) = R[2]$ .

2. Dually, for a cooperad  $\mathcal{C}$  cogenerated by  $V$  with co-relations  $Q$ , the quadratic dual operad  $\mathcal{C}^\perp$  is generated by  $V[-1]$  with relations given by the kernel

$$\text{Ker}(Q[-2] \rightarrow (V[-1] \circ V[-1])(3)).$$

### 3.1.3. Examples

The operads **COM**, **ASS**, **LIE** are quadratic. Their quadratic dual cooperads are given by the formulas

- $\text{COM}^\perp = (\text{LIE}\{-1\})^*$ ,
- $\text{ASS}^\perp = (\text{ASS}\{-1\})^*$ ,
- $\text{LIE}^\perp = (\text{COM}\{-1\})^*$ .

**3.1.4. Definition.** Let  $\mathcal{O}$  be a (graded) quadratic operad. A structure of  $\mathcal{O}_\infty$ -algebra on  $X \in \mathbf{Vectgr}$  is given by a differential on the cofree  $\mathcal{O}^\perp$ -coalgebra cogenerated by  $X$ .

3.1.5. The above definition gives rise to an operad  $\mathcal{O}_\infty$  in the category of complexes  $C(k)$ .

Let  $X$  have a structure of  $\mathcal{O}_\infty$ -algebra. The differential

$$(7) \quad Q : \mathbb{F}_{\mathcal{O}^\perp}^*(X) \rightarrow \mathbb{F}_{\mathcal{O}^\perp}^*(X)[1]$$

is defined uniquely by its composition with the projection onto the degree one component  $F_{\mathcal{O}^\perp}^{*1}(X) = X$ . Thus, the differential is given by the collection of maps

$$(8) \quad Q_i : \mathbb{F}_{\mathcal{O}^\perp}^{*i}(X) = (\mathcal{O}^\perp(i) \otimes X^{\otimes i})^{S_i} \rightarrow X[1].$$

in particular,  $d := Q_1$  defines a differential on  $X \in \mathbf{Vectgr}$ .

Define  $\mathcal{O}_\infty = \mathbb{T}(\mathcal{O}^\perp)$  to be the free graded operad generated by  $\mathcal{O}^\perp$ . The collection of maps  $Q_i$  from (8) defines an action of  $\mathcal{O}_\infty$  on  $X$ . One endows  $\mathcal{O}_\infty$  with a differential so that the condition  $Q^2 = 0$  is equivalent to the statement that the action of  $\mathcal{O}_\infty$  on  $(X, d = Q_1)$  respects the differentials.

3.1.6. **Example.** Let  $X$  be a complex endowed with a  $\mathcal{O}$ -algebra structure (dg  $\mathcal{O}$ -algebra). Define the differential  $Q$  on  $\mathbb{F}_{\mathcal{O}^\perp}^*(X)$  as follows.

$Q_1 : X \rightarrow X[1]$  is the differential of  $X$ .  $Q_2 : \mathcal{O}^\perp(2) \otimes X^{\otimes 2} \rightarrow X[1]$  is defined by the  $\mathcal{O}$ -algebra structure on  $X$  since  $\mathcal{O}^\perp(2) = \mathcal{O}(2)[1]$ .  $Q_i$  are defined to be zero for  $i > 2$ .

The condition  $Q^2 = 0$  can be easily verified. This means that any  $\mathcal{O}$ -algebra admits a canonical  $\mathcal{O}_\infty$ -algebra structure.

Example 3.1.6 shows there is a canonical map of operads in  $C(k)$

$$(9) \quad \mathcal{O}_\infty \rightarrow \mathcal{O}$$

(Here  $\mathcal{O}$  is supposed to have zero differential).

3.1.7. **Definition.** A quadratic operad  $\mathcal{O}$  is called Koszul if the natural map (9) is a quasi-isomorphism.

Let  $\mathcal{O}$  be a quadratic operad and let  $X$  be an  $\mathcal{O}_\infty$ -algebra (for instance, an  $\mathcal{O}$ -algebra). The homology of  $X$ ,  $H_{\mathcal{O}}(X)$ , is defined to be the homology of the complex  $(\mathbb{F}_{\mathcal{O}^\perp}^*(X), Q)$ .

If  $X = \mathbb{F}_{\mathcal{O}}(V)$  for a graded vector space  $V$ , one has a canonical map of complexes

$$(10) \quad (\mathbb{F}_{\mathcal{O}^\perp}^*(X), Q) \rightarrow V.$$

The following result can be used to prove koszulity of a quadratic operad.

3.1.8. **Theorem.** (cf. [GK], Thm. 4.2.5) A quadratic operad  $\mathcal{O}$  is Koszul iff for any graded vector space  $V$  the canonical map (10) is quasi-isomorphism.

Theorem 3.1.8 implies that the operads **COM**, **ASS**, **LIE** are Koszul.

## 4. DEFORMATIONS AND FORMALITY

4.1. **Intrinsic formality.** In this section  $\mathcal{O}$  is a fixed Koszul operad.

4.1.1. **Definition.** A  $\mathcal{O}_\infty$ -algebra  $X$  is called to be *formal* if there exists a pair of quasi-isomorphisms of  $\mathcal{O}_\infty$ -algebras  $X \leftarrow F \rightarrow H(X)$ .

4.1.2. **Definition.** A graded  $\mathcal{O}$ -algebra  $H$  is *intrinsically formal* if any  $\mathcal{O}_\infty$ -algebra  $X$  with  $H(X) = H$  is formal.

The aim of this section is to prove a criterion of intrinsic formality.

Let  $H$  be a graded  $\mathcal{O}$ -algebra and let  $\mathfrak{g}$  be the dg Lie algebra of coderivations of the corresponding dg  $\mathcal{O}^\perp$ -coalgebra  $(\mathbb{F}_{\mathcal{O}^\perp}^*(H), Q)$ . Since  $\mathbb{F}_{\mathcal{O}^\perp}^*(H)$  is cofree, any coderivation is uniquely defined by its composition with the projection onto  $H$ . Therefore,  $\mathfrak{g}$  considered as a graded vector space, is isomorphic to  $\text{Hom}(\mathbb{F}_{\mathcal{O}^\perp}^*(H), H)$ . We denote

$$\mathfrak{g}_{\geq 1} = \text{Hom}(\bigoplus_{i \geq 2} \mathbb{F}_{\mathcal{O}^\perp}^{*i}(H), H).$$

This is a dg Lie subalgebra of  $\mathfrak{g}$ .

4.1.3. **Theorem.** Suppose that the map  $H^1(\mathfrak{g}_{\geq 1}) \rightarrow H^1(\mathfrak{g})$  is zero. Then  $H$  is intrinsically formal.

4.2. **Proof of Theorem 4.1.3.** The following standard lemma results from the fact that  $\mathcal{O}_\infty$  is *cofibrant*.

4.2.1. **Lemma.** Let  $X$  be a  $\mathcal{O}_\infty$ -algebra. There exists a  $\mathcal{O}_\infty$ -algebra structure on  $H(X)$  so that  $X$  and  $H(X)$  are equivalent  $\mathcal{O}_\infty$ -algebras (i.e., there exists a pair of quasi-isomorphisms of  $\mathcal{O}_\infty$ -algebras  $X \leftarrow F \rightarrow H(X)$ ).

4.2.2. Let  $H$  be a graded  $\mathcal{O}$ -algebra. Let  $X$  be a  $\mathcal{O}_\infty$ -algebra so that  $H = H(X)$  as  $\mathcal{O}$ -algebras. Choose a  $\mathcal{O}_\infty$ -algebra structure on  $H$  guaranteed by Lemma 4.2.1. One has  $\mathcal{O}_\infty(2) = \mathcal{O}(2)$  and the  $\mathcal{O}$ -algebra structure on  $H$  is the restriction of the  $\mathcal{O}_\infty$ -algebra structure. To fix a notation, let the collection of maps

$$(11) \quad Q_n : \mathbb{F}_{\mathcal{O}^\perp}^{*n}(H) \rightarrow H[1],$$

$n \geq 2$  define the said  $\mathcal{O}_\infty$ -algebra structure on  $H$ . The  $\mathcal{O}$ -algebra structure on  $H$  is given by the collection  $\{Q_n^0\}$  with  $Q_2^0 = Q_2$ ;  $Q_i^0 = 0$  for  $i > 2$ .

**4.2.3. Lemma.** *Let  $\lambda \in k$ . Put  $Q_n^\lambda = \lambda^{n-2}Q_n$ . The collection  $\{Q_n^\lambda\}_{n \geq 1}$  defines a collection of  $\mathcal{O}_\infty$ -algebra structures on  $H$  parametrized by  $\lambda \in k$ . This gives the structure  $\{Q_n\}$  for  $\lambda = 1$  and  $\{Q_n^0\}$  for  $\lambda = 0$ .*

*Proof.* The only property we have to check to make sure that the collection  $\{Q_n^\lambda\}$  defines a  $\mathcal{O}_\infty$ -algebra structure, is the identity looking like

$$d(Q_n^\lambda) = P_n(Q_2^\lambda, \dots, Q_{n-1}^\lambda)$$

where  $P_n$  is a quadratic (non-commutative) polynomial. Since  $H$  has zero differential (this means  $Q_1 = 0$  in our notation) the left-hand side vanishes. The right hand side vanishes for  $\lambda = 1$  since the collection of  $Q_i$  does define a  $\mathcal{O}_\infty$ -action. Since the polynomials  $P_n$  are homogeneous, one has

$$P_n(Q_2^\lambda, \dots, Q_{n-1}^\lambda) = \lambda^{n-1} P_n(Q_2, \dots, Q_{n-1}).$$

This proves the claim.  $\square$

**4.2.4.** Put  $C = (\mathbb{F}_{\mathcal{O}^\perp}^*(H), Q^0)$ . This is a differential graded  $\mathcal{O}^\perp$ -coalgebra. The collection  $\{Q_n^\lambda\}$  defines a  $k[\lambda]$ -linear differential  $Q^\lambda$  on the  $\mathcal{O}^\perp$ -coalgebra  $C[\lambda]$ . We wish to construct an isomorphism

$$\theta : (C[\lambda], Q^0) \rightarrow (C[\lambda], Q^\lambda)$$

which is identity modulo  $\lambda$ .

The isomorphism  $\theta$  is uniquely defined by a collection of maps

$$\theta_n : \mathbb{F}_{\mathcal{O}^\perp}^{*n}(H) \rightarrow H[\lambda]$$

with  $\theta_1 = \text{id}_H$ . We will be looking for  $\theta$  satisfying the following property.

$$(12) \quad \theta_n = \phi_n \cdot \lambda^{n-1} \text{ for some } \phi_n : \mathbb{F}_{\mathcal{O}^\perp}^{*n}(H) \rightarrow H.$$

An automorphism  $\theta$  satisfying (12) is constructed in 4.2.6 below. Then, tensoring  $\theta$  by  $k[\lambda]/(\lambda - 1)$ , we get an isomorphism of dg  $\mathcal{O}^\perp$ -coalgebras

$$\overline{\theta} : (C, Q^0) \xrightarrow{\sim} (C, Q).$$

This will prove Theorem 4.1.3.

**4.2.5.** Define an action of the multiplicative group  $k^*$  on  $C[\lambda]$  by the formulas

$$\mu * x = \mu^n \cdot x \text{ for } x \in \mathbb{F}_{\mathcal{O}^\perp}^{*n}(H); \quad \mu * \lambda = \mu \cdot \lambda.$$

The differentials  $Q$  and  $Q^\lambda$  have both degree  $-1$  with respect to this action:

$$\mu * Q(\mu^{-1} * x) = \mu^{-1} \cdot Q(x); \quad \mu * Q^\lambda(\mu^{-1} * x) = \mu^{-1} \cdot Q^\lambda(x).$$

The condition (12) means that  $\theta$  has degree zero with respect to the defined action of  $k^*$ .

4.2.6. The map  $\theta$  will be constructed by induction.

Suppose we have constructed an isomorphism

$$\theta : (C[\lambda]/(\lambda^n), Q^0) \xrightarrow{\sim} (C[\lambda]/(\lambda^n), Q^\lambda)$$

satisfying the property  $\theta_k = \phi_k \cdot \lambda^{k-1}$  for some  $\phi_k : \mathbb{F}_{\mathcal{O}^\perp}^{*k}(H) \rightarrow H$  for all  $k$ . This means in particular that  $\theta_k = 0$  for  $k > n$ .

Our aim is to lift  $\theta$  to a map

$$\tilde{\theta} : (C[\lambda]/(\lambda^{n+1}), Q) \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q^\lambda)$$

such that its components  $\tilde{\theta}_k$  satisfy the same property.

First of all, we lift  $\theta$  to the isomorphism

$$\theta' : (C[\lambda]/(\lambda^{n+1}), Q') \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q^\lambda)$$

taking  $\theta'_k = \theta_k$  for all  $k$ , where  $Q'$  is some differential uniquely defined by the above formula. The differential  $Q'$  has also degree  $-1$ . Since  $Q'$  coincides with  $Q_0$  modulo  $\lambda^n$ , one has actually an equality  $Q'_k = Q_k^0$  for  $k \leq n+1$  and  $Q'_{n+2} = \lambda^n \cdot z$  for some  $z : \mathbb{F}_{\mathcal{O}^\perp}^{*n+2}(H) \rightarrow H$ . One easily observes that the element  $z$  considered as a derivation, is a cycle. Therefore, there is a derivation  $u \in \mathfrak{g}^0$ , such that  $z = du$ . This gives an isomorphism

$$\eta = \exp(\lambda^n \cdot u) : (C[\lambda]/(\lambda^{n+1}), Q) \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q')$$

which is identity modulo  $\lambda^n$ . The inductive step will be accomplished if we are able to find an isomorphism between  $(C[\lambda]/(\lambda^{n+1}), Q)$  and  $(C[\lambda]/(\lambda^{n+1}), Q')$  having degree zero.

The components  $\eta_k$  of  $\eta$  are divisible by  $\lambda^n$  for  $k > 1$ . An easy calculation shows that the collection  $\kappa_k : \mathbb{F}_{\mathcal{O}^\perp}^{*k}(H) \rightarrow H$  given by the formulas

$$\kappa_1 = \text{id}_H, \quad \kappa_{n+1} = \eta_{n+1}, \quad \kappa_i = 0 \text{ for } i \neq 1, n+1,$$

defines an isomorphism

$$\kappa : (C[\lambda]/(\lambda^{n+1}), Q) \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q').$$

The composition of  $\kappa$  with  $\theta'$  is the isomorphism  $\tilde{\theta}$  we were looking for.

The construction of isomorphism  $\theta$  satisfying (12), and, therefore, the proof of Theorem 4.1.3, is accomplished.

## 5. HOCHSCHILD COMPLEX

**5.1. Hochschild complex.** Let  $A$  be an associative  $k$ -algebra. Its Hochschild complex  $\mathcal{C} := C^*(A; A)$  has components defined by the formula

$$\mathcal{C}^n = C^n(A; A) = \text{Hom}(A^{\otimes n}, A), \quad n = 0, 1, \dots$$

The graded vector space  $\mathcal{C}$  admits a LIE{1}-algebra structure which comes from the identification of  $\mathcal{C}[1]$  with the collection of coderivations of the cofree coalgebra (with counit) cogenerated by  $A[1]$ .

An explicit formula for the Lie bracket is given in 5.5.3 below.

The multiplication  $\mu : A^{\otimes 2} \rightarrow A$  belongs to  $\mathcal{C}^2$ ; therefore the operator  $\text{ad } \mu$  has degree 1. An easy calculation shows that  $(\text{ad } \mu)^2 = 0$ ;  $\mathcal{C}$  endowed with the differential  $\text{ad } \mu$  becomes a dg LIE{1}-algebra.

**5.2.  $H(\mathcal{C})$  is a  $\mathcal{G}$ -algebra.** In order to prove Theorem 1.2 it would be enough to check that  $H = H(C^*(A; A))$  is intrinsically formal as a Lie algebra. This, however, is not true. Tamarkin's idea is to prove that  $H$  becomes intrinsically formal when it is considered as an algebra over an operad  $\mathcal{G}$  described below.

Define  $m : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$  by the formula

$$(13) \quad m(x \otimes y) = \mu \circ (x \boxtimes y)$$

where  $x \boxtimes y : A^{\otimes m+n} \rightarrow A^{\otimes 2}$  is defined to be the tensor product of the maps  $x : A^{\otimes m} \rightarrow A$  and  $y : A^{\otimes n} \rightarrow A$ .

The following lemma is due to M. Gerstenhaber (1964).

**5.2.1. Lemma.** *The map  $m$  induces a commutative associative multiplication on  $H(\mathcal{C})$ . The bracket on  $H(\mathcal{C})$  is a derivation with respect to  $m$ .*

**5.2.2. Definition.** Operad  $\mathcal{G}$  is the operad generated by the operations  $m \in \mathcal{G}(2)^0$ ,  $\ell \in \mathcal{G}(2)^{-1}$  satisfying the following identities:

- $m$  is commutative associative
- $\ell$  is Lie
- $\ell$  is a derivation with respect to  $m$ .

Lemma 5.2.1 above means that the cohomology  $H(\mathcal{C}(A; A))$  admits a natural  $\mathcal{G}$ -algebra structure.

**5.2.3.** The following construction assigns a  $\mathcal{G}$ -algebra to any Lie algebra  $\mathfrak{g}$ . Put  $X = \mathbb{F}_{\text{COM}}(\mathfrak{g}[-1]) = \bigoplus_{i>0} S^i(\mathfrak{g}[-1])$ . There is a unique LIE{1}-algebra structure on  $X$  extending that on  $\mathfrak{g}[-1]$  such that  $X$  becomes a  $\mathcal{G}$ -algebra. This is a *free  $\mathcal{G}$ -algebra generated by a Lie algebra  $\mathfrak{g}$* .

**5.2.4.** There is a twisted (=sheaf) version of the above construction. Let  $\mathfrak{g}$  be a Lie algebroid over a commutative algebra  $A$ . This mean that  $\mathfrak{g}$  is a Lie algebra,  $A$ -module, and a map of Lie algebras and  $A$ -modules  $\pi : \mathfrak{g} \rightarrow \text{Der}(A, A)$  is given so that

$$[f, ag] = a[f, g] + \pi(f)(a)g$$

for  $a \in A$ ,  $f, g \in \mathfrak{g}$ .

Then a  $\mathcal{G}$ -algebra structure on the  $A$ -symmetric algebra without unit  $S_A^{\geq 1}(\mathfrak{g}[-1])$  is naturally defined. If one defines  $A = S_A^0(\mathfrak{g}[-1])$  to commute with  $S_A^{\geq 1}(\mathfrak{g}[-1])$ , one obtains a  $\mathcal{G}$ -algebra structure on the  $A$ -symmetric algebra  $S_A(\mathfrak{g}[-1])$ .

**5.3. Koszulity.** The operad  $\mathcal{G}$  is obviously quadratic. The quadratic dual cooperad  $\mathcal{G}^\perp$  has as cogenerators elements  $\tilde{m}$ ,  $\tilde{\ell}$  of degrees  $-1$  and  $-2$  respectively. A simple calculation gives

### 5.3.1. Lemma.

$$\mathcal{G}^\perp = \mathcal{G}^*[2].$$

One has the following important

### 5.3.2. Proposition. ([GJ]) $\mathcal{G}$ is Koszul.

For an easy proof of this fact see 5.4.6.

Recall that koszulity of  $\mathcal{G}$  means that the natural map (9)

$$\mathcal{G}_\infty \rightarrow \mathcal{G}$$

is a quasi-isomorphism of operads. The operad  $\mathcal{G}_\infty$  is the operad for *homotopy Gerstenhaber algebras*.

Deformation theory approach to the Formality Theorem is based on the following version of *Deligne's conjecture*.

### 5.3.3. Theorem. There is a structure of $\mathcal{G}_\infty$ -algebra natural in $A$ on $C^*(A; A)$ inducing the described above $\mathcal{G}$ -algebra structure on $H(C^*(A; A))$ .

Theorem 5.3.3 will be proven in Sections 6 and 7. In this section we will deduce Formality Theorem 1.2 from Theorem 5.3.3.

**5.4. Calculation.** From now on  $A$  is a smooth commutative  $k$ -algebra. Our aim is to calculate the cohomology of  $H := H(C^*(A; A))$  and to make sure it vanishes when  $A$  is a polynomial algebra. This, together with Theorem 5.3.3, gives Formality Theorem.

The following classical result of Hochschild-Kostant-Rosenberg describes the cohomology of  $C^*(A; A)$ .

### 5.4.1. Lemma. $H = S_A(T_A[-1])$ where $T_A = \text{Der}(A, A)$ . The $\mathcal{G}$ -algebra structure on $H$ is defined as in 5.2.4.

Following 4.1.3, we have to calculate the dg Lie algebra of coderivations of the dg  $\mathcal{G}^\perp$ -coalgebra  $(\mathbb{F}_{\mathcal{G}^\perp}^*(H), Q)$  corresponding to  $H$ .

5.4.2. Note the following formula

$$(14) \quad \mathbb{F}_{\mathcal{G}^\perp}^*(X) = \mathbb{F}_{\text{COM}}^*(\mathbb{F}_{\text{LIE}}^*(X[1])[1])[-2]$$

which can be obtained using 5.3.1 from the formula dual to the following

$$(15) \quad \mathbb{F}_{\mathcal{G}}(X) = \mathbb{F}_{\text{COM}}(\mathbb{F}_{\text{LIE}\{1\}}(X)).$$

5.4.3. According to 4.1.3, we have to calculate the map  $H^1(\mathfrak{g}_{\geq 1}) \rightarrow H^1(\mathfrak{g})$  where

$$\mathfrak{g} = \text{Coder}(\mathbb{F}_{\mathcal{G}^\perp}^*(H)) = \text{Hom}(\mathbb{F}_{\mathcal{G}^\perp}^*(H), H) = \text{Hom}(\mathbb{F}_{\text{COM}}^*(\mathbb{F}_{\text{LIE}}^*(H[1])[1]), H[2])$$

with the differential induced by the differential  $Q$  of  $\mathbb{F}_{\mathcal{G}^\perp}^*(H)$ .

The differential  $Q$  of  $\mathbb{F}_{\mathcal{G}^\perp}^*(H)$  comes from the map  $\mathcal{G}(2) \otimes H^{\otimes 2} \rightarrow H$  describing the  $\mathcal{G}$ -algebra structure on  $H$ . Therefore,  $Q = Q_m + Q_\ell$  where  $Q_m$  is induced by the commutative multiplication  $m : H \otimes H \rightarrow H$ , and  $Q_\ell$  is induced by the bracket  $\ell : H \otimes H \rightarrow H[-1]$ . Since the defining relations on operations  $m$  and  $\ell$  in  $\mathcal{G}$  are homogeneous, one necessarily has

$$Q_m^2 = Q_\ell^2 = Q_m Q_\ell + Q_\ell Q_m = 0.$$

The total differential  $Q$  on  $\mathfrak{g}$  is also a sum of two differentials which will be denoted by  $Q_m$  and  $Q_\ell$ .

Any cofree coalgebra is naturally graded — see (6). Formula (14) gives rise to a bigrading on the cofree  $\mathcal{G}^\perp$ -coalgebra  $\mathbb{F}_{\mathcal{G}^\perp}^*(H)$  in which the  $(p, q)$ -component consists of the elements of **COM**-degree  $-p$  and total **LIE**-degree  $-q$ .

This defines a bigrading on  $\mathfrak{g}$  so that

$$\mathfrak{g}^{p\bullet} = \text{Hom}(\mathbb{F}_{\text{COM}}^{*1+p}(\mathbb{F}_{\text{LIE}}^{*\bullet}(H[1])[1]), H[2]).$$

Note that

$$(16) \quad \mathfrak{g}_{\geq 1} = \bigoplus_{(p,q) \neq (0,0)} \mathfrak{g}^{pq}.$$

The differentials  $Q_m$  and  $Q_\ell$  have degrees  $(0, 1)$  and  $(1, 0)$  with respect to this bigrading and  $\mathfrak{g}$  lives in the first quadrant. Therefore, one can use the spectral sequence argument to calculate the cohomology of  $\mathfrak{g}$ .

Let us calculate the first term  $E_1^{pq} = H^{pq}(\mathfrak{g}, Q_m)$ . To keep track of the differential  $Q_m$  in  $\mathfrak{g}$  it is convenient to present

$$\mathfrak{g}^{0q} = \text{Hom}(\mathbb{F}_{\text{LIE}}^{*1+q}(H[1])[1], H[2]) = \text{Hom}_H(\mathbb{F}_{\text{LIE}}^{*1+q}(H[1]) \otimes H[1], H[2])$$

and to identify  $\mathbb{F}_{\text{LIE}}^*(H[1]) \otimes H$  with the homological Harrison complex  $Z := \text{Harr}_*(H, H)$ .

Then one can see that  $(\mathfrak{g}^{0\bullet}, Q_m)$  coincides with  $\text{Hom}_H(Z[1], H[2])$  as a complex; moreover, for each  $p$  one has

$$(\mathfrak{g}^{p\bullet}, Q_m) = \text{Hom}_H(S_H^{1+p}(Z[1]), H[2]).$$

5.4.4. The considerations above hold for every graded  $\mathcal{G}$ -algebra  $H$  with unit. Now we will use the fact that  $H = H(C^*(A; A))$  where  $A$  is a smooth  $k$ -algebra.

Namely, according to Lemma 5.4.1,  $H$  is smooth as a graded commutative algebra. Therefore, there is a natural isomorphism

$$Z \xrightarrow{\sim} \Omega[1]$$

where  $\Omega = \Omega_{H/k}$  is the module of Kähler differentials. This implies that

$$(17) \quad E_1^{pq} = \begin{cases} \text{Hom}_H(S_H^{1+p}(\Omega[2]), H[2]), & q = 0 \\ 0, & q \neq 0. \end{cases}$$

Let us calculate  $\Omega_{H/k}$ . The sequence of smooth morphisms of graded commutative algebras

$$k \rightarrow A \rightarrow H = S_A(T_A[-1])$$

gives rise to an isomorphism

$$(18) \quad \Omega \xrightarrow{\sim} H \otimes_A \Omega_{A/k} \oplus \Omega_{H/A} = H \otimes_A (T_A[-1] \oplus T_A^*) = H \otimes_A \omega$$

where  $\omega = T_A[-1] \oplus T_A^*$ . Note that  $\omega$  is a finitely generated graded projective  $A$ -module.

The only non-vanishing cohomology in (17) can be rewritten as

$$(19) \quad E_1^{p0} = S_A^{1+p}(\omega[-1]) \otimes_A H[2] = S_H^{1+p}(\Omega[-1])[2]$$

since  $\omega[2]^* = \omega[-1]$ .

Note that  $E_1^{p0}$  embeds into

$$\mathfrak{g}^{p0} = \text{Hom}(\mathbb{F}_{\text{COM}}^{*1+p}(H[2]), H[2])$$

and the differential  $Q_\ell$  on the latter is defined by the Lie algebra structure on  $H[1]$ . This allows one to identify the the differential  $Q_2$  on (19) with the differential on the (shifted and truncated) de Rham complex of  $H$ .

5.4.5. Suppose now that  $A$  is a polynomial algebra over  $k$ . In this case de Rham complex of  $H$  is acyclic. Then the calculation in the previous subsection gives a quasi-isomorphism  $\mathfrak{g} \xrightarrow{\sim} H/k[1]$ . This implies that  $\mathfrak{g}$  has no cohomology coming from the cohomology of  $\mathfrak{g}_{\geq 1}$ .

5.4.6. *Remark.* A calculation similar to the above proves that  $\mathcal{G}$  is Koszul.

In fact, according to 3.1.8, one has to check that for each graded vector space  $V$  the natural map

$$V \rightarrow (\mathbb{F}_{\mathcal{G}^\perp}^*(\mathbb{F}_{\mathcal{G}}(V)), Q)$$

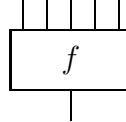
is a quasi-isomorphism.

Taking into account the formulas (14) and (15) and using, as in 5.4.3, the presentation of  $\mathbb{F}_{\mathcal{G}^\perp}^*(H)$  by a bicomplex, one easily obtains the result.

5.4.7. Now Theorem 1.2 have been proven modulo Theorem 5.3.3. In the end of this section we will describe the operad  $\mathcal{B}_\infty$  naturally acting on the Hochschild complex  $C(A; A)$  of any associative algebra  $A$ . This is the first step in the proof of Theorem 5.3.3 which is presented in Sections 6 and 7.

5.5. **Hochschild complex is a  $\mathcal{B}_\infty$ -algebra.** We shall now describe an operad which acts naturally on the Hochschild complex of any associative algebra. This operad is denoted  $\mathcal{B}_\infty$ . It has been invented by H.-I. Baues; its action on the Hochschild complex was defined in [GJ].

5.5.1. *Notation.* In this subsection  $A$  is any associative  $k$ -algebra and  $\mathcal{C} = C^*(A; A)$ . It is convenient to denote elements  $f \in \mathcal{C}^n$  as boxes having  $n$  hands and one leg like this:



5.5.2. *Basic operation.* Let  $f, g_1, \dots, g_n \in \mathcal{C}$ . Denote the brace  $f\{g_1, \dots, g_n\}$  by the following formula

$$(20) \quad f\{g_1, \dots, g_n\} = \sum_{\text{all possible insertions}} \begin{array}{c} | & | & | \\ g_1 & g_2 & g_n \\ | & | & | \\ \hline f \end{array}$$

The diagram shows a large box labeled 'f' at the bottom. Above it, three smaller boxes labeled  $g_1$ ,  $g_2$ , and  $g_n$  are arranged horizontally. Vertical lines connect the top of  $g_1$  to the top of  $f$ , the top of  $g_2$  to the top of  $f$ , and the top of  $g_n$  to the top of  $f$ . Vertical lines also connect the bottom of  $g_1$  to the bottom of  $f$ , the bottom of  $g_2$  to the bottom of  $f$ , and the bottom of  $g_n$  to the bottom of  $f$ . The labels  $g_1$ ,  $g_2$ , and  $g_n$  are placed inside their respective boxes.

Here the sum is taken over all possible order preserving insertions of legs of  $g_i$  into hands of  $f$ .

5.5.3. The Lie bracket on  $\mathcal{C}[1]$  is given explicitly, in terms of braces, by the formula

$$[f, g] = f\{g\} - (-1)^{|f||g|} g\{f\}.$$

5.5.4. **Definition.** A  $\mathcal{B}_\infty$ -algebra structure on a graded vector space  $X$  is given by a structure of dg bialgebra on  $\mathbb{F}_{\text{Ass}}^*(X[1])$  so that the coalgebra structure is the standard (cofree) one.

Let us check that  $\mathcal{B}_\infty$ -algebra structure is given by an operad (as usual, it will be denoted by  $\mathcal{B}_\infty$ ).

The dg bialgebra structure on  $\mathbb{F}_{\text{Ass}}^*(X[1])$  is given by the following data.

- a differential  $X[1]^{\otimes n} \rightarrow X[1]^{\otimes m}$  of degree 1. The differential is uniquely defined by its  $m = 1$  part. We denote its  $(n, 1)$ -components by  $m_n : X[1]^{\otimes n} \rightarrow X[2]$  (or, what is the same,  $m_n : X^{\otimes n} \rightarrow X[2 - n]$ ).
- a multiplication  $X[1]^{\otimes p} \otimes X[1]^{\otimes q} \rightarrow X[1]^{\otimes r}$  of degree 0 — it is also uniquely defined by its  $r = 1$  part. We denote the collection of  $r = 1$  multiplications by

$$m_{pq} : X[1]^{\otimes p} \otimes X[1]^{\otimes q} \rightarrow X[1]$$

or, what is the same,

$$m_{pq} : X^{\otimes p} \otimes X^{\otimes q} \rightarrow X[1 - p - q].$$

Therefore, the  $\mathcal{B}_\infty$ -algebra structure is given by a collection of operations  $m_n$ ,  $m_{pq}$  subject to some relations. This defines an operad  $\mathcal{B}_\infty$  as the one generated by  $m_n \in \mathcal{B}_\infty(n)^{2-n}$  and  $m_{pq} \in \mathcal{B}_\infty(p+q)^{1-p-q}$  subject to some relations.

5.5.5. **WARNING.** The operad  $\mathcal{B}_\infty$  is not obtained in any sense from a(ny) Koszul operad  $\mathcal{B}$ . Getzler and Jones are responsible for this notation.

5.5.6. *Action of  $\mathcal{B}_\infty$  on  $C^*(A; A)$ .* We have to define the action of the operations  $m_n$ ,  $m_{pq}$  on  $\mathcal{C} = C^*(A; A)$  and to check the compatibilities. Here it is.

- $m_1$  is the differential in  $\mathcal{C}$
- $m_2$  is the multiplication  $\mu$  defined by (13)
- $m_i = 0$  for  $i > 2$
- $m_{1k}(f \otimes g_1 \otimes \dots \otimes g_k) = f\{g_1, \dots, g_k\}$  where the brace operations are defined by formula (20)
- $m_{kl} = 0$  for  $k > 1$ .

One can directly check that the collection of operations  $m_n$ ,  $m_{pq}$  defined above gives rise to a  $\mathcal{B}_\infty$ -algebra structure on  $\mathcal{C}$ .

## 6. BETWEEN $\mathcal{G}$ AND $\mathcal{G}_\infty$

In this section we present an operad  $\tilde{\mathcal{B}}$  lying between  $\mathcal{G}$  and  $\mathcal{G}_\infty$  — it admits a pair of maps

$$\mathcal{G}_\infty \rightarrow \tilde{\mathcal{B}}, \quad \tilde{\mathcal{B}} \rightarrow \mathcal{G}$$

so that the composition is the obvious map  $\mathcal{G}_\infty \rightarrow \mathcal{G}$ .

In the next section we will prove, using Etingof-Kazhdan theorem on quantization of Lie bialgebras [EK], that the operad  $\tilde{\mathcal{B}}$  is isomorphic to the operad  $\mathcal{B}_\infty$  acting on the Hochschild complex of any associative algebra by 5.5.6. This will yield Theorem 5.3.3 and, therefore, Theorem 1.2.

6.1.  **$\tilde{\mathcal{B}}$ -algebras.** A  $\tilde{\mathcal{B}}$ -algebra structure on a graded vector space  $X$  is a dg Lie bialgebra structure on  $\mathbb{F}_{\text{LIE}}^*(X[1])$  extending the standard free Lie coalgebra structure.

The Lie bracket on a Lie bialgebra  $\mathbb{F}_{\text{LIE}}^*(X[1])$  is defined by its corestriction to the cogenerators  $X[1]$ . Therefore, it is given by a collection of maps

$$\ell_{mn} : \mathbb{F}_{\text{LIE}}^{*m}(X[1]) \otimes \mathbb{F}_{\text{LIE}}^{*n}(X[1]) \rightarrow X[1]$$

satisfying a collection of quadratic identities. The differential on  $\mathbb{F}_{\text{LIE}}^*(X[1])$  is also defined by its corestriction to the cogenerators. This amounts to a collection

$$d_n : \mathbb{F}_{\text{LIE}}^{*n}(X[1]) \rightarrow X[2]$$

satisfying some more quadratic identities — the one saying that  $d^2 = 0$  and the other that the  $d$  is the derivation of the Lie algebra structure given by  $\ell_{mn}$ .

In particular, one has  $d_1^2 = 0$  and this endows  $X$  with a structure of complex. The obvious maps  $X[1] \rightarrow \mathbb{F}_{\text{LIE}}^*(X[1]) \rightarrow X[1]$  are maps of complexes.

Since the  $\tilde{\mathcal{B}}$ -structure on  $X$  is given by a collection of operations subject to some relations, there is an operad in the category of complexes which will be called in the sequel  $\tilde{\mathcal{B}}$  such that  $\tilde{\mathcal{B}}$ -algebras are just algebras over  $\tilde{\mathcal{B}}$ .

6.2. A map  $\mathcal{O} \rightarrow \mathcal{O}'$  of operad endows a  $\mathcal{O}'$ -algebra with a canonical  $\mathcal{O}$ -algebra structure. The converse is also obviously true — in order to define a map of operads it is enough to endow any  $\mathcal{O}'$ -algebra with a canonical  $\mathcal{O}$ -algebra structure.

Let us construct a map  $\tilde{\mathcal{B}} \rightarrow \mathcal{G}$ . For this we have to define canonically a  $\tilde{\mathcal{B}}$ -algebra structure on each  $\mathcal{G}$ -algebra  $X$ . Recall that a  $\mathcal{G}$ -algebra  $X$  is endowed with a commutative multiplication  $m : X^{\otimes 2} \rightarrow X$  and a Lie bracket  $l : X[1]^{\otimes 2} \rightarrow X[1]$ . The Harrison complex of the commutative algebra  $(X, m)$  is given by a differential on  $\mathbb{F}_{\text{LIE}}^*(X[1])$ . The Lie algebra structure on  $X[1]$  can be uniquely extended to  $\mathbb{F}_{\text{LIE}}^*(X[1])$  to get a Lie bialgebra. The Harrison differential will be a derivation with respect to the Lie algebra structure, so this construction defines a dg Lie bialgebra structure on  $\mathbb{F}_{\text{LIE}}^*(X[1])$ .

The construction is obviously canonical and yields a morphism of operads  $\tilde{\mathcal{B}} \rightarrow \mathcal{G}$ .

6.3. Let now construct a map  $\mathcal{G}_\infty \rightarrow \tilde{\mathcal{B}}$ .

Let  $X$  be a  $\tilde{\mathcal{B}}$ -algebra. This means that a dg Lie bialgebra structure on  $\mathfrak{g} = \mathbb{F}_{\text{LIE}}^*(X[1])$  is given. In particular,  $\mathfrak{g}$  is a dg Lie algebra and this defines a differential on  $\mathbb{F}_{\text{COM}}^*(\mathfrak{g}[1])$ . The latter complex is by formula (14) just  $\mathbb{F}_{\mathcal{G}^\perp}^*(X)[2]$ . Differential on it gives a  $\mathcal{G}_\infty$ -structure on  $X$ .

Thus the map  $\mathcal{G}_\infty \rightarrow \tilde{\mathcal{B}}$  is constructed.

## 7. EQUIVALENCE OF $\tilde{\mathcal{B}}$ WITH $\mathcal{B}_\infty$

In this section we prove that the operads  $\tilde{\mathcal{B}}$  and  $\mathcal{B}_\infty$  are isomorphic. Note that the isomorphism is obtained using Etingof-Kazhdan theorems [EK] on quantization of Lie bialgebras. The isomorphism will depend on the choice of associator, as in [EK].

7.1. For some technical reasons, it is more convenient to use coalgebras over  $\tilde{\mathcal{B}}$  and  $\mathcal{B}_\infty$  instead of algebras. Our aim is to prove that any  $\tilde{\mathcal{B}}$ -coalgebra admits a natural  $\mathcal{B}_\infty$ -coalgebra structure and vice versa.

Note

7.1.1. **Lemma.**  *$\tilde{\mathcal{B}}$ -coagebra structure on a graded vector space  $X$  is given by a structure of dg Lie bialgebra on*

$$\widehat{\mathbb{F}}_{\text{LIE}}(X[1]) = \prod_{n=0}^{\infty} \mathbb{F}_{\text{LIE}}^n(X[1]).$$

7.1.2. **Lemma.**  *$\mathcal{B}_\infty$ -coagebra structure on a graded vector space  $X$  is given by a structure of dg bialgebra on*

$$\widehat{\mathbb{F}}_{\text{ASS}}(X[1]) = \prod_{n=0}^{\infty} X[1]^{\otimes n}.$$

Now we wish to use [EK] in order to pass from one structure above to the other. The idea is the following. One can interpret completions of free algebras  $\mathbb{F}_{\text{LIE}}(V)$  and  $\mathbb{F}_{\text{ASS}}(V)$  as equivariant  $k[[h]]$ -algebras  $\mathbb{F}_{\text{LIE}}(V)[[h]]$  and  $\mathbb{F}_{\text{ASS}}(V)[[h]]$ . This is the situation Etingof-Kazhdan theory applies.

7.2. **Etingof-Kazhdan theory.** Let  $\text{Locc}(k)$  be the category of local complete  $k$ -algebras with residue field  $k$ .

Let  $\mathcal{A}$  be an abelian  $k$ -linear tensor category. For each  $R \in \text{Locc}(k)$  we denote by  $\mathcal{A}(R)$  the category with the same objects as  $\mathcal{A}$  and with the morphisms defined by the formula

$$\text{Hom}_{\mathcal{A}(R)}(X, Y) = \text{Hom}_{\mathcal{A}}(X, Y) \otimes R.$$

The object of  $\mathcal{A}(R)$  corresponding to an object  $X \in \mathcal{A}$  is denoted  $X_R$ . In the other direction, for  $Y \in \mathcal{A}(R)$  we write  $\overline{Y}$  for the corresponding object of  $\mathcal{A}$ . The assignments  $X \mapsto X_R$  and  $Y \mapsto \overline{Y}$  define a pair of functors between  $\mathcal{A}$  and  $\mathcal{A}(R)$ .

Let  $\text{LBA}_0(R)$  be the category of Lie bialgebras  $(\mathfrak{g}, [\ , \ ], \delta)$  in  $\mathcal{A}(R)$  whose cobracket  $\delta$  vanishes modulo the maximal ideal  $\mathfrak{m}$  of  $R$ . Let  $\text{HA}_0$  denote the category of Hopf algebras in  $\mathcal{A}_R$  whose reduction modulo  $\mathfrak{m}$  is isomorphic to the enveloping algebra of a Lie algebra in  $\mathcal{A}$ .

The following theorem can be found in [EK].

7.2.1. **Theorem.** *There is an equivalence of categories*

$$(21) \quad Q : \mathbf{LBA}_0(R) \rightarrow \mathbf{HA}_0(R)$$

*satisfying the following properties (see also explanations below)*

1.  $\overline{Q(\mathfrak{g})} = U(\overline{\mathfrak{g}})$
2.  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  measures the deviation of the coproduct in  $Q(\mathfrak{g})$  from being cocommutative.
3.  $Q$  is given by universal formulas.

7.2.2. The second property of the functor  $Q$  mentioned in Theorem 7.2.1 means the following. Denote by  $i : \mathfrak{g} \rightarrow Q(\mathfrak{g})$  the image under the functor  $\otimes R$  of the obvious embedding  $\overline{\mathfrak{g}} \rightarrow U(\overline{\mathfrak{g}})$ . The property 2 claims that the map from  $\mathfrak{g}$  to  $Q(\mathfrak{g}) \otimes Q(\mathfrak{g})$  given by the difference

$$(\Delta - \Delta') \circ i - (i \otimes i) \circ \delta$$

vanishes modulo  $\mathfrak{m}^2$ .

Here  $\Delta$  is the coproduct in  $Q(\mathfrak{g})$  and  $\Delta'$  is the coproduct composed with the commutativity constraint.

7.2.3. The third property means the following. As an object of  $\mathcal{A}(R)$ ,  $Q(\mathfrak{g})$  is just the symmetric algebra  $S(\mathfrak{g}) = \bigoplus S^n(\mathfrak{g})$ . Therefore, the Hopf algebra structure on  $Q(\mathfrak{g})$  is given by a collection of maps  $m_{pqr} : S^p(\mathfrak{g}) \otimes S^q(\mathfrak{g}) \rightarrow S^r(\mathfrak{g})$  and  $\Delta_{pqr} : S^p(\mathfrak{g}) \rightarrow S^q(\mathfrak{g}) \otimes S^r(\mathfrak{g})$ . Universality condition means that the maps  $m_{pqr}$ ,  $\Delta_{pqr}$  are described as universal polynomials on the bracket and cobracket in  $\mathfrak{g}$ .

7.2.4. It is convenient to define  $\mathbf{LBA}_0$  to be the category of pairs  $(R, \mathfrak{g})$  where  $R \in \mathbf{Locc}(k)$  and  $\mathfrak{g} \in \mathbf{LBA}_0(R)$ . In the same fashion one defines the category  $\mathbf{HA}_0$ . Since the functors  $Q : \mathbf{LBA}_0(R) \rightarrow \mathbf{HA}_0(R)$  are given by the universal formulas, they form a functor  $Q : \mathbf{LBA}_0 \rightarrow \mathbf{HA}_0$  which is also an equivalence of categories. Reduction modulo the maximal ideal defines a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{LBA}_0 & \xrightarrow{Q} & \mathbf{HA}_0 \\ \downarrow & & \downarrow \\ \mathbf{Lie}(k) & \xrightarrow{U} & \mathbf{HA}_0(k) \end{array}$$

where  $\mathbf{Lie}(k)$  is the category of Lie algebras over  $k$  and  $U$  is the enveloping algebra functor.

7.2.5. The equivalence of categories  $Q : \mathbf{LBA}_0 \rightarrow \mathbf{HA}_0$  gives rise to an equivalence  $Q^G : \mathbf{LBA}_0^G \rightarrow \mathbf{HA}_0^G$  between the categories of objects endowed with a  $G$ -action,  $G$  being a group.

Let now  $\mathcal{A}$  be the category of complexes of  $k$ -modules. Let  $G = k^*$  be the multiplicative group,  $R = k[[h]]$ . Let  $k^*$  act on  $R$  by the formula  $\lambda(h) = \lambda^{-1}h$ . Let  $V \in \mathcal{A}$ . Let  $k^*$  act on  $V$  by the formula  $\lambda(v) = \lambda \cdot v$ . This action extends to a  $k^*$ -action on  $\mathbb{F}_{\text{LIE}}(V)$  and  $\mathbb{F}_{\text{ASS}}(V)$ , as well as to an action on  $\mathbb{F}_{\text{LIE}}(V)[[h]]$  and  $\mathbb{F}_{\text{ASS}}(V)[[h]]$ .

Theorem 7.2.1 implies the following

7.2.6. **Corollary.** *The functor  $Q$  establishes an equivalence between the following categories:*

1. *Lie bialgebras  $(R, \mathfrak{g}) \in \mathbf{LBA}_0$  endowed with a  $k^*$ -action compatible with the specified above  $k^*$ -action on  $R = k[[h]]$  and on  $\overline{\mathfrak{g}} = \mathbb{F}_{\text{LIE}}(V)$ .*
2. *Associative bialgebras  $(R, H) \in \mathbf{HA}_0$  endowed with a  $k^*$ -action compatible with the specified above action on  $R = k[[h]]$  and on  $\overline{H} = \mathbb{F}_{\text{ASS}}(V)$ .*

7.3. **Theorem.** *There exists an isomorphism between the operads  $\widehat{\mathcal{B}}$  and  $\mathcal{B}_\infty$ .*

Theorem 7.3 is proven in 7.3.1–7.3.3 below.

7.3.1. Put  $\mathfrak{g} = \mathbb{F}_{\text{LIE}}(V)$ . A  $k[[h]]$ -Lie bialgebra structure on  $\mathfrak{g}[[h]]$  is given by a collection of maps

$$\delta_{pq}^r : V \rightarrow \mathbb{F}_{\text{LIE}}^p(V) \otimes \mathbb{F}_{\text{LIE}}^q(V)$$

such that the cobracket  $\delta : \mathfrak{g}[[h]] \rightarrow \mathfrak{g}[[h]] \otimes \mathfrak{g}[[h]]$  restricted to  $\mathfrak{g}$  is given by the formula

$$\mathfrak{g} = \sum_{p,q,r} \delta_{pq}^r \cdot h^r.$$

Define an action of  $k^*$  on  $\mathfrak{g}[[h]]$  as in 7.2.5. The cobracket  $\delta$  of  $\mathfrak{g}[[h]]$  is equivariant if and only if it satisfies the property

$$(22) \quad \delta_{pq}^r = 0 \text{ for } r \neq p + q - 1.$$

One can easily identify dg Lie bialgebra structures on  $\mathfrak{g}[[h]]$  satisfying (22) with dg Lie bialgebra structures on  $\widehat{\mathfrak{g}}$ .

7.3.2. Similarly, dg bialgebra structures on  $\widehat{\mathbb{F}}_{\text{ASS}}(V)$  can be identified with equivariant bialgebra structures on  $\mathbb{F}_{\text{ASS}}(V)[[h]]$ .

7.3.3. We use Corollary 7.2.6 of the equivalence  $Q$  from Etingof-Kazhdan Theorem 7.2.1.

Let  $X$  be a complex,  $V = X[1]$ .  $\tilde{\mathcal{B}}$ -coalgebra structure on  $X$  is given by a structure of dg Lie bialgebra on  $\widehat{\mathfrak{g}} = \widehat{\mathbb{F}}_{\text{LIE}}(V)$  which is the same as an equivariant dg Lie bialgebra structure on  $\mathfrak{g}[[h]]$ .

According to Corollary 7.2.6, this defines canonically an equivariant dg Hopf algebra  $(H, m, \Delta) \in \text{HA}_0$ .

The canonical map  $i : V[[h]] \rightarrow H$  given by the composition

$$i : V \rightarrow \mathfrak{g} \rightarrow \overline{H}$$

induces an algebra homomorphism  $F(i) : \mathbb{F}_{\text{ASS}}(V)[[h]] \rightarrow H$ . It is isomorphism since its reduction modulo  $h$   $\overline{F(i)}$  is the identity map. This defines canonically an equivariant bialgebra structure on  $\mathbb{F}_{\text{ASS}}(V)[[h]]$  which is the same as a dg bialgebra structure on  $\widehat{\mathbb{F}}_{\text{ASS}}(V)$ . Theorem is proven.

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